Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(04)00050-4
J. Appl. Maths Mechs, Vol. 68, No. 3, pp. 369–385, 2004

0021–8928/$—see front matter

PERIODIC MOTIONS OF A CLOSE TO DYNAMICALLY SYMMETRICAL SATELLITE IN THE NEIGHBOURHOOD OF A CONICAL PRECESSION†

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(*Received* 14 *October* 2003)

The motions of a close to dynamically symmetrical satellite in a circular orbit, that is, of a rigid body in a central Newtonian gravitational field, are considered. The periodic motions, generated from the conical precession of a dynamically symmetrical satellite, are constructed in the unperturbed problem. A rigorous, non-linear analysis of the stability of these motions is carried out. In the unperturbed problem, one of the coordinates, the angle of natural rotation of the satellite, is cyclic and the system of differential equations describing the motions of the perturbed problem is close to the system with the cyclic coordinate. The resonant case, when the ratio of one of the frequencies of small oscillations of the reduced system in the neighbourhood of a stable equilibrium to the frequency of the change in the cyclic coordinate is close to an integer and the case when there is no resonance are investigated. Previously obtained [1] results of an investigation of the periodic motions of autonomous Hamiltonian systems with two degrees of freedom are extended to the case of a system with three degrees of freedom being considered here, when the above-mentioned resonance is present. When there is no such resonance, the cases of parametric resonance, of third- and fourth-order resonance and, also, the general non-resonant case are distinguished. Results for the stability of non-autonomous Hamiltonian systems with two degrees of freedom in the case of resonances [2] and, also, the results of KAM-theory (in the general non-resonant case) [3] are used.

The stability of the conical precession of a dynamically symmetrical satellite in a circular orbit has been investigated in [4–7]. In the case of a weakly elliptic orbit [8] and in the case of a close to dynamically symmetrical satellite in a circular orbit [9], the periodic motions of a satellite have been found (in the form of power series in a small parameter) and their stability has been investigated in the linear approximation. The periodic motions of a dynamically symmetrical satellite in a weakly elliptic orbit have been investigated for the case of resonance in forced oscillations, when one of the frequencies of the small oscillations of the satellite is close to the average motion of its centre of mass [10]. The $2\pi p$-periodic motions of a dynamically symmetrical satellite in an elliptic orbit, generated from the $2\pi p/q$-periodic motions in a circular orbit, are constructed and their stability is analysed in the linear approximation in [11].

1. FORMULATION OF THE PROBLEM

We consider the motion of a satellite, that is, of a rigid body moving in a circular orbit in a central Newtonian gravitational field. Suppose $GXYZ$ is the orbital system of coordinates with origin at the centre of mass G of the satellite. Its axes GX, GY and GZ are directed along the transversal, along the binormal to the orbit and along the radius vector of the centre of mass, respectively. We will associated a system of coordinates $Gxyz$ with the satellite, the axes of which are directed along its principal central axes of inertia. The orientation of the system of coordinates $Gxyz$ with respect to $GXYZ$ will be specified using the Euler angles ψ, θ and φ.

†*Prikl. Mat. Mekh.* Vol. 68, No. 3, pp. 414–432, 2004.

The motion of the satellite about the centre of mass is described by canonical differential equations with Hamiltonian

$$\begin{aligned} H = {} & \left(\frac{A}{B}\cos^2\varphi + \sin^2\varphi\right)\frac{p_\psi^2}{2\sin^2\theta} + \frac{1}{2}\left(\frac{A}{B}\sin^2\varphi + \cos^2\varphi\right)p_\theta^2 + \\ & + \frac{1}{2}\left[\left(\frac{A}{B}\cos^2\varphi + \sin^2\varphi\right)\operatorname{ctg}^2\theta + \frac{A}{C}\right]p_\varphi^2 - \left(\frac{A}{B} - 1\right)\frac{\sin\varphi\cos\varphi}{\sin\theta}p_\psi p_\theta - \\ & - \left(\frac{A}{B}\cos^2\varphi + \sin^2\varphi\right)\frac{\operatorname{ctg}\theta}{\sin\theta}p_\psi p_\varphi - \left(\frac{A}{B} - 1\right)\sin\varphi\cos\varphi\operatorname{ctg}\theta p_\theta p_\varphi - \cos\psi\operatorname{ctg}\theta p_\psi - \\ & - \sin\psi p_\theta + \frac{\cos\psi}{\sin\theta}p_\varphi + \frac{3}{2}\left[\left(\frac{B}{A}\cos^2\varphi + \sin^2\varphi\right)\sin^2\theta + \frac{C}{A}\cos^2\theta\right] \end{aligned} \tag{1.1}$$

where A, B and C are the principal central moments of inertia of the satellite and p_ψ, p_θ, p_φ are the momenta corresponding to the coordinates ψ, θ and φ, which have been made dimensionless using the factor $A\omega_0$, where ω_0 is the mean motion of the centre of mass. The variable $\tau = \omega_0 t$ is taken as the independent variable.

Suppose the moments of inertia A and B of the satellite are similar. Then, on introducing the small parameter $\varepsilon = (A - B)/B$ $(0 < \varepsilon \ll 1)$ and, also, the parameter $\alpha = C/A$ $(0 < \alpha \leqslant 2)$, Hamiltonian (1.1) can be written in the form

$$\begin{aligned} H = {} & H^{(0)} + \varepsilon H^{(1)} \\ H^{(0)} = {} & \frac{p_\psi^2}{2\sin^2\theta} + \frac{p_\theta^2}{2} + \frac{1}{2}\left(\operatorname{ctg}^2\theta + \frac{1}{\alpha}\right)p_\varphi^2 - \frac{\operatorname{ctg}\theta}{\sin\theta}p_\psi p_\varphi - \\ & - \cos\psi\operatorname{ctg}\theta p_\psi - \sin\psi p_\theta + \frac{\cos\psi}{\sin\theta}p_\varphi + \frac{3}{2}(\alpha - 1)\cos^2\theta \\ H^{(1)} = {} & \frac{\cos^2\varphi}{2\sin^2\theta}p_\psi^2 + \frac{1}{2}\sin^2\varphi p_\theta^2 + \frac{1}{2}\cos^2\varphi\operatorname{ctg}^2\theta p_\varphi^2 - \frac{\sin\varphi\cos\varphi}{\sin\theta}p_\psi p_\theta - \\ & - \frac{\cos^2\varphi\operatorname{ctg}\theta}{\sin\theta}p_\psi p_\varphi - \sin\varphi\cos\varphi\operatorname{ctg}\theta p_\theta p_\varphi - \frac{3}{2(1+\varepsilon)}\cos^2\varphi\sin^2\theta \end{aligned} \tag{1.2}$$

where $H^{(0)}$ is the unperturbed Hamiltonian corresponding to the motion of a dynamically symmetrical $(A = B)$ satellite. The φ coordinate in the system with Hamiltonian $H^{(0)}$ is cyclic and this means that $p_\varphi = p_{\varphi_0} = \text{const}$. The perturbing part $\varepsilon H^{(1)}$ of Hamiltonian (1.2) contains the φ coordinate and is periodic with respect to it with a period equal to π. A system with Hamiltonian (1.2) is therefore close to a system with a cyclic coordinate.

We will now consider the particular motion of the unperturbed system, described by the relations

$$\begin{aligned} & \theta = \theta_0 = \arcsin\frac{p_{\varphi_0}}{3\alpha - 4}, \quad p_\theta = p_{\theta_0} = 0 \\ & \psi = \psi_0 = 0, \quad p_\psi = p_{\psi_0} = 3(\alpha - 1)\sin\theta_0\cos\theta_0 \\ & \varphi(\tau) = \Omega\tau + \varphi(0), \quad \Omega = \frac{4(\alpha - 1)}{\alpha}\sin\theta_0 \end{aligned}$$

and which corresponds to a conical precession of a dynamically symmetrical satellite. In the case of a conical precession, the axis of the satellite is perpendicular to the velocity vector of the centre of mass and at an angle θ_0 to the velocity vector of the centre of mass. At the same time, the satellite is rotating about its axis with angular velocity Ω.

In Fig. 1, the domain I $(0 < \alpha < 1)$, where sufficient conditions for the stability of conical precession are satisfied, and the domain II, where only the necessary conditions for stability are satisfied, have been separated out in the plane of the parameters α, θ_0 $(0 \leqslant \alpha \leqslant 2, 0 \leqslant \theta_0 \leqslant \pi/2)$. Domain II is defined by the relations

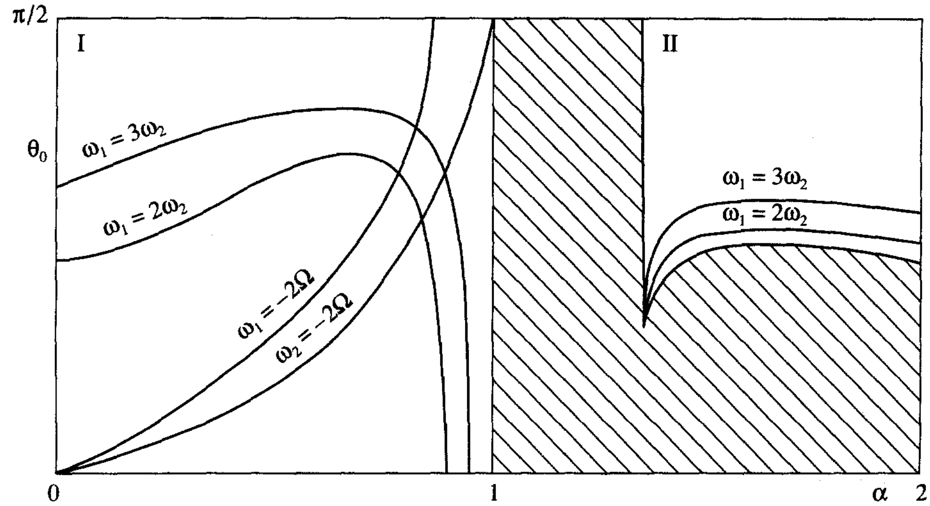

Fig. 1

$$\alpha \geq \frac{4}{3},\ \sin^2\theta_0 \geq \frac{18\alpha^2 - 27\alpha + 8 + 2(3\alpha - 2)\sqrt{(3\alpha - 1)(3\alpha - 4)}}{27\alpha^2(\alpha - 1)}$$

The frequencies ω_1 and ω_2 ($\omega_1 > \omega_2$) of small oscillations of the reduced system with two degrees of freedom in the neighbourhood of the stable equilibrium position in domains I and II are determined from the equation [7]

$$\omega^4 - [7 - 6\alpha - 9\alpha(1 - \alpha)\sin^2\theta_0]\omega^2 + 3\cos^2\theta_0(4 - 3\alpha)(1 - \alpha) = 0$$

The conical precession is unstable in the hatched domain in Fig. 1.

We will now take the conical precession of the dynamically symmetrical satellite (for values of the parameters α and θ_0 from domains I and II) as the unperturbed motion and consider the motions of a close to dynamically symmetrical ($\varepsilon \neq 0$) satellite in its neighbourhood. We will construct the periodic motions of the satellite which are close to its conical precession in the unperturbed problem and investigate their stability.

2. TRANSFORMATION OF THE HAMILTONIAN

In (1.2), we put

$$\theta = \theta_0 + q_1,\quad p_\theta = p_{\theta_0} + p_1,\quad \psi = \psi_0 + q_2,\quad p_\psi = p_{\psi_0} + p_2,\quad \varphi = q,\quad p_\varphi = p_{\varphi_0} + P$$

The functions $H^{(0)}$ and $H^{(1)}$ can be written in the form

$$\begin{aligned} H^{(0)} &= H_2^{(0)} + H_3^{(0)} + H_4^{(0)} + O_5 \\ H^{(1)} &= H_0^{(1)} + H_1^{(1)} + H_2^{(1)} + H_3^{(1)} + H_4^{(1)} + O_5 \end{aligned} \tag{2.1}$$

where $H_k^{(1)}$ ($i = 0, 1$) is the set of terms of the kth order with respect to the quantities q_j, p_j ($j = 1, 2$) and $|P|^{1/2}$ with constant coefficients (when $i = 0$) or (when $i = 1$) π-periodic coefficients with respect to q with harmonics $\cos 2q$ and $\sin 2q$; O_5 is the set of terms of no lower than the fifth order with respect to the same quantities.

The functions $H_k^{(0)}$ have the form $H_k^{(0)} = H_{k0}^{(0)} = H_{k1}^{(0)}$, where

$$H_{k0}^{(0)} = \sum_{\nu_1+\nu_2+\mu_1+\mu_2=k} h_{\nu_1\nu_2\mu_1\mu_2} q_1^{\nu_1} q_2^{\nu_2} p_1^{\mu_1} p_2^{\mu_2}, \quad k = 2, 3, 4$$

$$H_{21}^{(0)} = \Omega P, \quad H_{31}^{(0)} = (m_1 q_1 + m_2 p_2)P, \quad H_{41}^{(0)} = (l_1 q_1^2 + l_2 q_1 p_2 + l_3 q_2^2)P + nP^2$$

$$h_{2000} = \frac{1}{2}[9(\alpha-1)(\alpha\sin^2\theta_0 - 1) + \operatorname{ctg}^2\theta_0], \quad h_{1001} = 3\alpha - 2 - \frac{1}{\sin^2\theta_0}$$

$$h_{0200} = \frac{1}{2}[1 - 3(\alpha-1)\sin^2\theta_0], \quad h_{0110} = -1, \quad h_{0020} = \frac{1}{2}, \quad h_{0002} = \frac{1}{2\sin^2\theta_0}$$

$$h_{3000} = -\frac{9}{2}\sin\theta_0\cos\theta_0\alpha^2 - \frac{3}{2}\operatorname{ctg}\theta_0(7\cos^2\theta_0 - 10)\alpha - \frac{\cos\theta_0}{\sin^3\theta_0}(6\cos^4\theta_0 - 16\cos^2\theta_0 + 11)$$

$$h_{2001} = -\frac{9}{2}\operatorname{ctg}\theta_0\alpha + \frac{2\cos\theta_0}{\sin^3\theta_0}(2\sin^2\theta_0 + 1)$$

$$h_{0201} = -h_{1200} = \frac{1}{2}\operatorname{ctg}\theta_0, \quad h_{1002} = -\frac{\cos\theta_0}{\sin^3\theta_0} \tag{2.2}$$

$$h_{4000} = \frac{3}{8}(7\cos^2\theta_0 + 8)\alpha^2 + \frac{3(11\cos^4\theta_0 - 6\cos^2\theta_0 - 21)}{8\sin^2\theta_0}\alpha +$$

$$+\frac{9\cos^6\theta_0 - 23\cos^4\theta_0 - 7\cos^2\theta_0 + 30}{6\sin^4\theta_0}, \quad h_{2200} = -\frac{3}{4}\alpha + \frac{\sin^2\theta_0 + 1}{2\sin^2\theta_0}$$

$$h_{3001} = \frac{7\cos^2\theta_0 + 5}{2\sin^2\theta_0}\alpha - \frac{10\sin^2\theta_0\cos^2\theta_0 + 9}{3\sin^4\theta_0}, \quad h_{2002} = \frac{2\cos^2\theta_0 + 1}{2\sin^4\theta_0}$$

$$h_{1201} = -\frac{1}{2\sin^2\theta_0}, \quad h_{0400} = \frac{1}{8}\sin^2\theta_0\alpha + \frac{1}{8}\cos^2\theta_0 - \frac{1}{6}, \quad h_{0310} = \frac{1}{6}$$

$$n = \frac{1}{2}\left(\operatorname{ctg}^2\theta_0 + \frac{1}{\alpha}\right), \quad m_1 = \frac{\cos\theta_0}{\sin^2\theta_0}[1 - 3\sin^2\theta_0(\alpha-1)], \quad m_2 = -\frac{\cos\theta_0}{\sin^2\theta_0}$$

$$l_1 = \frac{3\sin^2\theta_0(\cos^2\theta_0 + 2)\alpha + 3\cos^4\theta_0 - 7}{2\sin^3\theta_0}, \quad l_2 = \frac{\cos^2\theta_0 + 1}{\sin^3\theta_0}, \quad l_3 = -\frac{1}{2\sin\theta_0}$$

The coefficients $h_{\nu_1\nu_2\mu_1\mu_2}$ which have not been written out are equal to zero.
For the function $H^{(1)}$ in (2.1) we only give the terms $H_0^{(1)}$ and $H_1^{(1)}$:

$$H_0^{(1)} = a\cos^2\varphi, \quad H_1^{(1)} = b\cos^2\varphi q_1 + c\sin\varphi\cos\varphi p_1 + d\cos^2\varphi p_2$$

$$a = 2\cos^2\theta_0 - \frac{3}{2}, \quad b = -\operatorname{ctg}\theta_0(1 + 6\sin^2\theta_0 - 3\alpha\sin^2\theta_0) \tag{2.3}$$

$$c = (7 - 6\alpha)\cos\theta_0, \quad d = \operatorname{ctg}\theta_0$$

We now carry out a number of canonical replacements of the variables which simplify the structure of the Hamiltonian H. First, using the linear substitution

$$
\begin{aligned}
q_1 &= n_{11}q_1^* + n_{12}q_2^*, \quad q_2 = n_{23}p_1^* + n_{24}p_2^*, \quad q = q^* \\
p_1 &= n_{33}p_1^* + n_{34}p_2^*, \quad p_2 = n_{41}q_1^* + n_{42}q_2^*, \quad P = P^* \\
n_{11} &= k_1 n_{23}, \quad n_{12} = \pm k_2 n_{24}, \quad n_{23} = (\omega_1 A_1)^{-1/2}, \quad n_{24} = (\pm\omega_2 A_2)^{-1/2} \\
n_{33} &= (1 + k_1\omega_1)n_{23}, \quad n_{34} = (1 + k_2\omega_2)n_{24} \\
n_{41} &= [k_1 - \sin^2\theta_0[\omega_1 + k_1(3\alpha - 2)]]n_{23} \\
n_{42} &= \pm[k_2 - \sin^2\theta_0[\omega_2 + k_2(3\alpha - 2)]]n_{24} \\
k_i &= \frac{3(1-\alpha) - \omega_i^2}{(3\alpha - 2)\omega_i}, \quad A_i = k_i^2 + 3(1-\alpha)\frac{\sin^2\theta_0}{\omega_i^2}; \quad i = 1, 2
\end{aligned}
\tag{2.4}
$$

we reduce the quadratic part of $H_2^{(0)}$ to the normal form

$$
H_2^{(0)*} = \frac{1}{2}\omega_1(q_1^{*2} + p_1^{*2}) \pm \frac{1}{2}\omega_2(q_2^{*2} + p_2^{*2}) + \Omega P^* \tag{2.5}
$$

In relations (2.4) and (2.5), the upper sign refers to the domain I and the lower sign to the domain II; it is assumed that $\alpha \neq 2/3$. In the case when $\alpha = 2/3$, one of the quantities k_i is not defined and, in the case, the quantities n_{ij} are calculated using the formulae

$$
n_{11} = n_{33}^{-1} = n_{41} = -\omega_1^{-1/2}, \quad n_{12} = n_{23} = 0, \quad n_{24} = n_{34} = -n_{42}^{-1} = \sin^{-1}\theta_0 \tag{2.6}
$$

for $0 < \theta_0 < \pi/4$ when $\omega_1 = \sqrt{2}\cos\theta_0$, $\omega_2 = 1$ and, using the formulae

$$
n_{12} = n_{34}^{-1} = n_{42} = -\omega_2^{-1/2}, \quad n_{11} = n_{24} = 0, \quad n_{23} = n_{33} = -n_{41}^{-1} = \sin^{-1}\theta_0 \tag{2.7}
$$

for $\pi/4 < \theta_0 < \pi/2$ when $\omega_1 = 1$, $\omega_2 = \sqrt{2}\cos\theta_0$. If, however, $\alpha = 2/3$, $\theta_0 = \pi/4$, then we have $\omega_1 = \omega_2 = 1$. We shall not discuss this case any further.

As a result of transformations (2.4), the terms $H_{k0}^{(0)}$ $(k = 3, 4)$ take the form

$$
H_{k0}^{(0)*} = \sum_{\nu_1+\nu_2+\mu_1+\mu_2=k} h^*_{\nu_1\nu_2\mu_1\mu_2} q_1^{*\nu_1} q_2^{*\nu_2} p_1^{*\mu_1} p_2^{*\mu_2}
$$

We shall not give the explicit form of the coefficients $h^*_{\nu_1\nu_2\mu_1\mu_2}$ here.

Suppose there are no third- and fourth-order resonance relations between the frequencies ω_1 and ω_2, that is $\omega_1 \neq 2\omega_2$ and $\omega_1 \neq 3\omega_2$. The resonance curves $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_1$ in domains I and II are shown in Fig. 1. For points outside these curves, it is possible to construct a transformation of the Birkhoff type which is close to a canonical identity transformation and has the form

$$
q_i^* = Q_i + \ldots, \quad p_i^* = P_i + \ldots, \quad i = 1, 2, \quad q^* = Q + \ldots, \quad P^* = P^* \tag{2.8}
$$

and which normalizes the unperturbed Hamiltonian $H^{(0)*}$ up to terms of the fourth order inclusive. This transformation was obtained using the *Deprit-Hori* method. Because of its complexity, it is not presented here. The Hamiltonian $H^{(0)*}$ takes the form

$$
\begin{aligned}
\tilde{H}^{(0)} &= \frac{1}{2}\omega_1 S_1^2 \pm \frac{1}{2}\omega_2 S_2^2 + \Omega P^* + \frac{1}{4}c_{11}S_1^4 + \frac{1}{4}c_{12}S_1^2S_2^2 + \frac{1}{4}c_{22}S_2^4 + \\
&+ \frac{1}{2}c_{13}S_1^2P^* + \frac{1}{2}c_{23}S_2^2P^* + c_{33}P^{*2} + O_5, \quad S_i^2 = Q_i^2 + P_i^2, \quad i = 1, 2
\end{aligned}
\tag{2.9}
$$

where c_{ij} are constant coefficients which are calculated using the formulae

$$
\begin{aligned}
c_{11} &= \frac{1}{2}(3h^*_{4000} + h^*_{2020} + 3h^*_{0040}) - \frac{3}{4\omega_1}(5h^{*2}_{3000} + 2h^*_{3000}h^*_{1020} + h^{*2}_{1020}) + \\
&+ \frac{1}{4\omega_2(\omega_2^2 - 4\omega_1^2)}[(h^{*2}_{2100} + h^{*2}_{0120})(8\omega_1^2 - 3\omega_2^2) + \\
&+ 2h^*_{2100}h^*_{0120}(8\omega_1^2 - \omega_2^2) + 4h^*_{1011}(h^*_{0120} - h^*_{2100})\omega_1\omega_2 - h^{*2}_{1011}\omega_2^2] \\
c_{22} &= \frac{1}{2}(3h^*_{0400} + h^*_{0202} + 3h^*_{0004}) - \frac{3}{4\omega_2}(5h^{*2}_{0300} + 2h^*_{0300}h^*_{0102} + h^{*2}_{0102}) + \\
&+ \frac{1}{4\omega_1(\omega_1^2 - 4\omega_2^2)}[(h^{*2}_{1200} + h^{*2}_{1002})(8\omega_2^2 - 3\omega_1^2) + \\
&+ 2h^*_{1200}h^*_{1002}(8\omega_2^2 - \omega_1^2) + 4h^*_{0111}(h^*_{1002} - h^*_{1200})\omega_1\omega_2 - h^{*2}_{0111}\omega_1^2] \\
c_{12} &= h^*_{2200} + h^*_{2002} + h^*_{0220} + h^*_{0022} - \frac{1}{\omega_1}(3h^*_{3000} + h^*_{1020})(h^*_{1200} + h^*_{1002}) - \\
&- \frac{1}{\omega_2}(3h^*_{0300} + h^*_{0102})(h^*_{2100} + h^*_{0120}) + \frac{2}{\omega_2^2 - 4\omega_1^2}[((h^*_{2100} - h^*_{0120})^2 + h^{*2}_{1011})\omega_1 + \\
&+ h^*_{1011}(h^*_{2100} - h^*_{0120})\omega_2] + \frac{2}{\omega_1^2 - 4\omega_2^2}[((h^*_{1200} - h^*_{1002})^2 + h^{*2}_{0111})\omega_2 + h^*_{0111}(h^*_{1200} - h^*_{1002})\omega_1] \\
c_{13} &= L_1 + L_3 - \frac{M_1}{\omega_1}(3h^*_{3000} + h^*_{1020}) - \frac{M_2}{\omega_2}(h^*_{2100} + h^*_{0120}) \\
c_{23} &= L_2 + L_4 - \frac{M_1}{\omega_1}(h^*_{1200} + h^*_{1002}) - \frac{M_2}{\omega_2}(3h^*_{0300} + h^*_{0102}) \\
c_{33} &= n - \frac{1}{2}\left(\frac{M_1^2}{\omega_1} + \frac{M_2^2}{\omega_2}\right) \\
M_i &= m_1 n_{1i} + m_2 n_{4i}, \quad L_i = n_{1i}(l_1 n_{1i} + l_2 n_{4i}), \quad i = 1, 2, \quad L_j = l_3 n_{2j}^2, \quad j = 3, 4
\end{aligned}
\tag{2.10}
$$

As a result of transformations (2.4) and (2.8), the function $H^{(1)}$ in the perturbing part of the Hamiltonian changes, but its structure remains the same as in (2.1).

3. PERIODIC MOTIONS OF A SATELLITE WHEN THERE ARE NO RESONANCES $\omega_1 \approx -2N\Omega$, $\omega_2 \approx -2N\Omega$

3.1. *Isoenergetic reduction*

In order to construct the periodic solutions, we will first consider the motions of a system on an isoenergetic level. Using the energy integral $H = \varepsilon\Omega h = \text{const}$, we change to a reduced, non-autonomous Hamiltonian system with two degrees of freedom and an independent variable Q. Here, the function

$$
\begin{aligned}
K &= \frac{1}{2}\frac{\omega_1}{\Omega}S_1^2 \pm \frac{1}{2}\frac{\omega_2}{\Omega}S_2^2 + \frac{1}{4}\hat{c}_{11}S_1^4 + \frac{1}{4}\hat{c}_{12}S_1^2S_2^2 + \frac{1}{4}\hat{c}_{22}S_2^4 + \\
&+ \varepsilon(\hat{H}_1 + \hat{H}_2 + \hat{H}_3 + \hat{H}_4) + O_5 + O(\varepsilon^2)
\end{aligned}
\tag{3.1}
$$

$$
\hat{c}_{ii} = c_{ii} - \frac{c_{i3}\omega_i}{\Omega} + \frac{c_{33}\omega_i^2}{\Omega^2}, \quad i = 1, 2, \quad \hat{c}_{12} = c_{12} - \frac{c_{13}\omega_2 + c_{23}\omega_1}{\Omega} + \frac{2c_{33}\omega_1\omega_2}{\Omega^2}
$$

where $\hat{H}_k$ is the set of kth order terms in S_i $(i = 1, 2)$, will play the role of the Hamiltonian. The function (3.1) is π-periodic with respect to Q and, generally speaking, contains all the even harmonics of Q. In particular, $\hat{H}_1$ has the form (the coefficients a, b, c and d are defined in (2.3))

$$\hat{H}_1 = a_1\cos^2 QQ_1 + a_2\cos^2 QQ_2 + b_1\sin 2QP_1 + b_2\sin 2QP_2$$
$$a_i = n_{1i}b + n_{4i}d, \quad b_i = \frac{1}{2}n_{3,2+i}c + a\frac{M_i}{\omega_i}, \quad i = 1, 2 \tag{3.2}$$

In the subsequent investigation it is necessary to distinguish between the resonant case, when the ratio of the frequencies ω_1/Ω or ω_2/Ω of the initial Hamiltonian system with three degrees of freedom is close to an even integer and the case when there is no resonance. The above-mentioned resonance is equivalent to the existence of resonance in the forced oscillations in the reduced, non-autonomous Hamiltonian system with two degrees of freedom with Hamiltonian (3.1). Resonance curves $\omega_i = -2N\Omega$ ($i = 1, 2$) exist in domain I (where $\Omega < 0$) for all $N = 1, 2, 3, \ldots$. The curves $\omega_1 = -2\Omega$ and $\omega_2 = -2\Omega$ are shown in Fig. 1. The resonant cases $\omega_i \simeq -2\Omega$ ($i = 1, 2$) will be investigated in Section 4. The cases of resonance $\omega_i \simeq -2N\Omega$, when $N \geqslant 2$, require that terms of the order of ε^2 and higher are taken into account in the Hamiltonian and will not be considered.

Calculations show that resonance relations of the form $\omega_i \simeq 2N\Omega$ ($i = 1, 2$) are not realized in domain II ($\Omega > 0$).

3.2. The periodic solution when there are no resonances $\omega_i \simeq -2N\Omega$ ($i = 1, 2$). Geometrical interpretation
Suppose there are no resonances of the form $\omega_i \simeq -2N\Omega$ ($i = 1, 2$) in the system, that is, the points (α, θ_0) do not belong to the curves $\omega_i = -2N\Omega$ ($i = 1, 2, N = 1, 2, 3, \ldots$) from domain I and their small neighbourhoods or lie in domain II. Following the Poincaré method, a unique solution of the system with Hamiltonian (3.1), which is π-periodic with respect to Q and analytic with respect to ε, can be constructed, which has the form

$$Q_i = Q_i^*(Q) = \varepsilon\left[-\frac{a_i}{2\omega_i} + \frac{1}{2}\chi_i^{ab}\cos 2Q\right] + O(\varepsilon^2), \quad P_i = P_i^*(Q) = \varepsilon\chi_i^{ba}\sin 2Q + O(\varepsilon^2)$$
$$\chi_i^{ab} = \frac{a_i\omega_i - 4b_i\Omega}{4\Omega^2 - \omega_i^2}, \quad \chi_i^{ba} = \frac{b_i\omega_i - a_i\Omega}{4\Omega^2 - \omega_i^2}, \quad i = 1, 2 \tag{3.3}$$

for the points of domain I; for the points of domain II, ω_2 must be replaced by $-\omega_2$.

From relations (3.3) and the energy integral $H = \varepsilon\Omega h$, we obtain a solution for the quantity P^*, which is π-periodic in Q (the coefficient a is defined in (2.3))

$$P^* = J^*(Q) = \varepsilon\left[h - \frac{a\cos^2 Q}{\Omega}\right] + O(\varepsilon^2) \tag{3.4}$$

Relations (3.3) and (3.4) specify a one-parameter family of solutions, which is π-periodic in Q and analytic with respect to ε, of a system with three degrees of freedom with Hamiltonian $\tilde{H} = \tilde{H}^{(0)} + \varepsilon\tilde{H}^{(1)}$, the unperturbed part $\tilde{H}^{(0)}$ of which is defined by formula (2.9). The energy constant h serves as the parameter.

In the initial variables, the following family of motions of the satellite, which are π-periodic in φ.

$$\theta = \theta_0 + \varepsilon(A_1 + A_2\cos 2\varphi) + O(\varepsilon^2), \quad \psi = \varepsilon B_1\sin 2\varphi + O(\varepsilon^2)$$
$$A_1 = -n_{11}\frac{a_1 + 2M_1h}{2\omega_1} - n_{12}\frac{a_2 + 2M_2h}{2\omega_2}, \quad A_2 = \frac{1}{2}n_{11}\chi_1^{ab} + \frac{1}{2}n_{12}\chi_2^{ab} \tag{3.5}$$
$$B_1 = n_{23}\chi_1^{ba} + n_{24}\chi_2^{ba}$$

corresponds to this family of solutions.

Here, the change in the variable φ is described by the equation

$$\frac{d\varphi}{d\tau} = \Omega + G(\varphi)$$
$$G(\varphi) = (m_1n_{11} + m_2n_{41})\left(Q_1^* - \frac{M_1}{\omega_1}J^*\right) +$$
$$+ (m_1n_{12} + m_2n_{42})\left(Q_2^* - \frac{M_2}{\omega_2}J^*\right) + 2nJ^* - \varepsilon\frac{\cos^2\theta_0}{\sin\theta_0}\cos^2\varphi + O(\varepsilon^2)$$

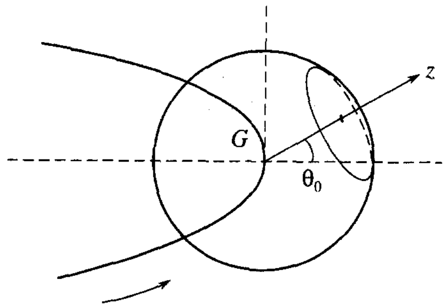

Fig. 2

The period of the solution (3.5) with respect to the "time" τ is equal to $T = 2\pi/\Omega_1$, where $\Omega_1 = \Omega + \bar{G}$ and $\bar{G}$ is the mean value of the function $G(\varphi)$ over a period π.

Relations (3.5) (if the terms $O(\varepsilon^2)$ are neglected) correspond to such a motion of the satellite when the unit vector of its axis describes a spatial curve on a sphere of unit radius with its centre at the centre of mass G, the projection of which onto the plane $GX'Y'$ of the orbital system of coordinates, which is perpendicular to the axis of the satellite in the unperturbed motion, is an ellipse (Fig. 2). The equation of the ellipse will have the form (the GX' axis coincides with GX and the GY' axis lies in the GXY plane and makes and angle θ_0 with the GY axis)

$$\frac{X'^2}{(\varepsilon B_1 \sin\theta_0)^2} + \frac{(Y' - \varepsilon A_1)^2}{(\varepsilon A_2)^2} = 1$$

The semi-axes of the ellipse have a length of the order of ε and its centre is displaced relative to the origin of the coordinates G by an amount of the order of ε.

3.3. The stability of the periodic solution

In order to solve the problem of the stability of the periodic solution (3.3), (3.4) in the Hamiltonian $\tilde{H}$ (with an unperturbed part (2.9)), using a canonical transformation of the form [12]

$$\begin{gathered} Q_i = Q_i^*(Q) + x_i, \quad P_i = P_i^*(Q) + y_i, \quad i = 1, 2, \quad Q = \varphi_3 \\ P^* = J^*(Q) + x_1 \frac{dP_1^*}{dQ} - y_1 \frac{dQ_1^*}{dQ} + x_2 \frac{dP_2^*}{dQ} - y_2 \frac{dQ_2^*}{dQ} + R_3 \end{gathered} \tag{3.6}$$

we introduce the perturbations of the variables Q_i, P_i ($i = 1, 2$) and P^* relative to their values for the periodic motion and subsequently change to the "polar" coordinates φ_i, R_i ($i = 1, 2$) using the formulae

$$x_i = \sqrt{2R_i}\sin\varphi_i, \quad y_i = \sqrt{2R_i}\cos\varphi_i$$

The perturbed Hamiltonian then takes the form

$$\Gamma = (\omega_1 R_1 \pm \omega_2 R_2 + \Omega R_3 + \varepsilon\Gamma_2^{(1)}) + \varepsilon\Gamma_3^{(1)} + \left(\sum_{i, j = 1;\, i \le j}^{3} c_{ij} R_i R_j + \varepsilon\Gamma_4^{(1)} \right) + O(\varepsilon^2) \tag{3.7}$$

where $\Gamma_k^{(1)}$ ($k = 2, 3, 4$) are forms of the kth power with respect to the quantities $|R_i|^{1/2}$ ($i = 1, 2, 3$) with coefficients which are π-periodic with respect to φ_3.

The case of parametric resonance. Suppose the parameters α and θ_0 from the domains I and II are such that the frequencies ω_1, ω_2 and Ω are linked by one of the relations $2\omega_i = 2N|\Omega|$ ($i = 1$ or $i = 2$), $\omega_1 \pm \omega_2 = 2N|\Omega|$. There is then parametric resonance in the system. Calculations show that the above-mentioned relations are not realized in domain II. At the same time, in domain I, the corresponding resonance curves (RC) exist for all values of N. The resonance curves (RC) $2\omega_1 = -2\Omega$, $2\omega_2 = -2\Omega$ and $\omega_1 + \omega_2 = -2\Omega$ (labelled with the pairs of numbers 2;0, 0;2 and 1;1 respectively) are shown in Fig. 3 for the case when $N = 1$. When $\varepsilon \ne 0$, a domain of parametric resonance is created from each point of these resonance curves (there are two surfaces in the three-dimensional space of the parameters α, θ_0 and ε which, when $\varepsilon = 0$, emerge from each resonance curve and specify the domains of parametric resonance). In the case of the points of the resonance curves $\omega_1 - \omega_1 = -2N\Omega$ from domain I, stability occurs when account is taken of terms in the Hamiltonian no higher than the first order in R_i.

Outside these resonance curves, the form of $\Gamma_2^{(1)}$ in the Hamiltonian (3.7) can be simplified by retaining just the "secular" terms in it. We then obtain

$$\begin{aligned} \Gamma^* &= (\lambda_1 R_1 \pm \lambda_2 R_2 + \Omega^* R_3) + \varepsilon\Gamma_3^{(1)} + (\textstyle\sum c_{ij} R_i R_j + \varepsilon\Gamma_4^{(1)*}) + O(\varepsilon^2) \\ \lambda_i &= \omega_i + O(\varepsilon) = \text{const}, \quad i = 1, 2, \quad \Omega^* = \Omega + O(\varepsilon) = \text{const} \end{aligned} \tag{3.8}$$

The cases of third and fourth-order resonances. We will now consider the cases of third- and fourth-order resonances. The corresponding resonance curves are specified by the equations $k_1\lambda_1 + k_2\lambda_2 = 2N|\Omega^*|$ ($N = 1, 2, \ldots$), where k_1 and k_2 are integers which satisfy the relation $|k_1| + |k_2| = l$ ($l = 3$ or $l = 4$).

Stability occurs in the corresponding finite approximation in the resonance curves from domains I and II, for which the relations $k_1k_2 < 0$ and $k_1k_2 > 0$ are satisfied respectively (only the resonance curves $\lambda_1 + 2\lambda_2 = 2N\Omega^*$, $\lambda_1 + 3\lambda_2 = 2N\Omega^*$ of the above-mentioned form exist in domain II).

Instability can only be observed in the resonance curves for which $k_1k_2 > 0$ in domain I and $k_1k_2 < 0$ in domain II (resonance curves only exist when $2\lambda_1 - \lambda_2 = 2N\Omega^*$, $4\lambda_2 = 2N\Omega^*$, $2\lambda_1 - 2\lambda_2 = 2N\Omega^*$ in the latter domain). We shall confine our consideration to the case when $N = 1$. Resonance effects when $N > 1$ manifest themselves in terms of the order of ε^N. Resonance curves of the form being considered are shown in Figs 3 and 4 for the case when $N = 1$. Each resonance curve is labelled with the corresponding pair of numbers k_1; k_2.

The periodic motion being considered is unstable for points of the third-order resonance curve if, in Hamiltonian (3.8), the coefficient in the term with the corresponding resonance harmonic in the form $\Gamma_3^{(1)}$ is non-zero [2]. Calculations show that the condition for instability is violated (the corresponding coefficient vanishes) for points of the resonance curves with the abscissae

$$0.153\ldots,\quad 0.661\ldots,\quad 0.796\ldots,\quad 0.897\ldots\quad (\text{RC } 3\lambda_2 = -2\Omega^*)$$

$$0.512\ldots,\quad 0.672\ldots,\quad 0.823\ldots,\quad 0.889\ldots\quad (\text{RC } \lambda_1 + 2\lambda_2 = -2\Omega^*)$$

$$0.145\ldots,\quad 0.643\ldots,\quad 0.820\ldots\quad (\text{RC } 2\lambda_1 + \lambda_2 = -2\Omega^*)$$

$$0.258\ldots,\quad 0.648\ldots\quad (\text{RC } 3\lambda_1 = -2\Omega^*)$$

$$1.361\ldots,\quad 1.518\ldots\quad (\text{RC } 2\lambda_1 - \lambda_2 = 2\Omega^* \text{ from domain II})$$

Suppose now that there are no resonances of up to the third order inclusive in the system and, at the same time, the point (α, θ_0) belongs to one of the fourth-order resonance curves. The form $\Gamma_3^{(1)}$ in Hamiltonian (3.8) can then be eliminated and the fourth-power terms simplified, taking account of the resonance which exists. Since the resonance components in the fourth-power terms are of the order of magnitude of ε while the coefficients c_{ij} are of the order of unity, then, as a rule, stability occurs in the fourth-order resonance curves when terms of no higher than the second order in R_i are taken into account in the Hamiltonian [2]. The points on the resonance curves, for which the coefficients $\hat{c}_{ij}$ in the fourth-order terms of the Hamiltonian of the reduced system with two degrees of freedom, corresponding to the system with Hamiltonian (3.8), satisfy the relations (see formulae (3.1) for the expressions for the coefficients $\hat{c}_{ij}$)

$$\begin{aligned} \hat{c}_{11} &= 0 \ (\text{for RC } 4\lambda_1 = -2\Omega^*) \\ 9\hat{c}_{11} + 3\hat{c}_{12} + \hat{c}_{22} &= 0 \ (\text{for RC } 3\lambda_1 + \lambda_2 = -2\Omega^*) \\ \hat{c}_{11} + \hat{c}_{12} + \hat{c}_{22} &= 0 \ (\text{for RC } 2\lambda_1 + 2\lambda_2 = -2\Omega^*) \\ \hat{c}_{11} + 3\hat{c}_{12} + 9\hat{c}_{22} &= 0 \ (\text{for RC } \lambda_1 + 3\lambda_2 = -2\Omega^*) \\ \hat{c}_{22} &= 0 \ (\text{for RC } 4\lambda_2 = -2\Omega^*) \end{aligned} \tag{3.9}$$

are an exception.

For the resonance curves (RC) $2\lambda_1 - 2\lambda_2 = 2\Omega^*$ and $4\lambda_2 = 2\Omega^*$ from domain II we have relations which are analogous to the third and fifth equations in (3.9).

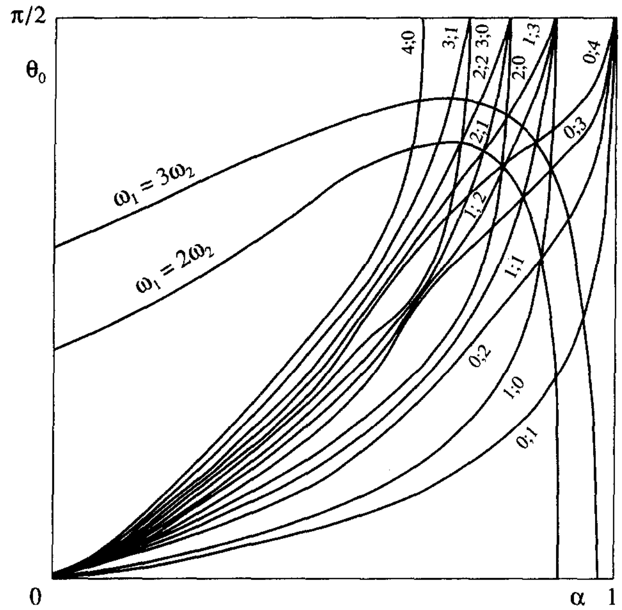

Fig. 3

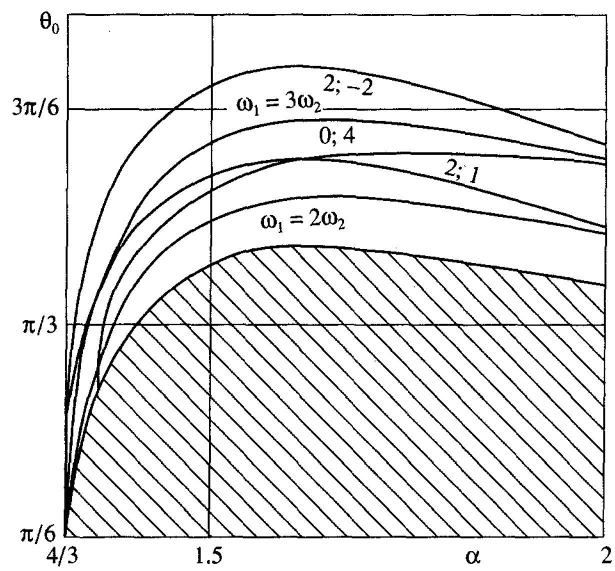

Fig. 4

Calculations show that the first relation of (3.9) is not realized and that the remaining relations in (3.9) hold at points with the abscissae 0.717 … ; 0.576 … , 0.632 … , 0.761 … ; 0.533 … , 0.673 … ; 0.397 … respectively. In domain II, the corresponding relation for the resonance curve (RC) $4\lambda_2 = -2\Omega^*$ is not realized while, for the resonance curve $2\lambda_1 - 2\lambda_2 = 2\Omega^*$, it holds at the point with abscissa 1.391 … .

Calculations were carried out separately using formulae (2.6) and (2.7) for points of the third- and fourth-order resonance curves when $\alpha = 2/3$. It was shown that, in the case of a third-order resonance, there is instability at these points (the resonance coefficients are non-zero) apart from at a point lying on the resonance curve $3\lambda_1 = -2\Omega^*$ where the corresponding resonance coefficient vanishes. Relations (3.9) and the analogous relations for domain II are not realized in the case of points of all the fourth-order resonance curves with abscissae $\alpha = 2/3$, that is, stability occurs in the final approximation.

The general non-resonant case. Finally, suppose there are no resonances of up to the fourth order inclusive. Then, Hamiltonian (3.8), which has been normalized up to terms of the fourth power, has the form

$$\Gamma = \Gamma^* + O(\varepsilon), \quad \Gamma^* = \lambda_1 R_1 \pm \lambda_2 R_2 + \Omega^* R_3 + \sum c_{ij}^* R_i R_j$$

$$c_{ij}^* = c_{ij} + O(\varepsilon) = \text{const}$$

The periodic motion being considered will be orbitally stable for the majority of initial conditions if the conditions [3]

$$D_3 = \det\left\|\frac{\partial^2\Gamma^*}{\partial R_i \partial R_j}\right\| \neq 0, \quad \text{or} \quad D_3 = \det\left\|\begin{matrix} \dfrac{\partial^2\Gamma^*}{\partial R_i \partial R_j} & \dfrac{\partial \Gamma^*}{\partial R_i} \\ \dfrac{\partial \Gamma^*}{\partial R_j} & 0 \end{matrix}\right\| \neq 0 \tag{3.10}$$

are satisfied.

The supplement to the above-mentioned majority of initial conditions is of the order of $O(\sqrt{\varepsilon})$ [3].

Calculations showed that $D_3 < 0$ and $D_4 > 0$ in domain II, and hence orbital stability occurs for all points of domain II outside the resonance curves for the majority of initial conditions. At the same time, resonance curves exist in domain I in which $D_3 = 0$ and $D_4 = 0$. The condition $D_3 = D_4 = 0$ is satisfied at the two points (0.604 ... ; 1.370 ...) and (0.817 ... ; 0.704 ...) of domain I. Hence, in the general non-resonant case, the periodic motion of the satellite being considered is orbitally stable (for the majority of initial conditions) both in domain II as well as in domain I (with the exception, perhaps, of the two points which have been indicated).

4. PERIODIC MOTIONS OF A SATELLITE IN THE CASE OF THE RESONANCES $\omega_I \approx -2\Omega (i = 1, 2)$

We will now construct the periodic motions of a satellite for values of the parameters α and θ_0 which belong to the resonance curves $\omega_1 = -2\Omega$ and $\omega_2 = -2\Omega$ or small neighbourhoods of these curves (see Fig. 1 and, also, Fig. 3, where these resonance curves are labelled with the pairs of numbers 1;0 and 0;1).

The theory of the resonant periodic motions of autonomous Hamiltonian systems with two degrees of freedom, which are close to systems with a cyclic coordinate, has been developed earlier [1]. We will now extend these results to the case of a system with three degrees of freedom which is being considered here.

In the Hamiltonian $\tilde{H}$ (with an unperturbed part (2.9)), we put

$$Q_i = \varepsilon^{1/3} Q_i^*, \quad P_i = \varepsilon^{1/3} P_i^*, \quad i = 1, 2, \quad P^* = \varepsilon^{2/3} J_3, \quad Q = \psi_3$$

and then change to the "polar" coordinates ψ_i, J_i $(i = 1, 2)$ using the formulae

$$Q_i^* = \sqrt{2J_i}\sin\psi_i, \quad P_i^* = \sqrt{2J_i}\cos\psi_i$$

The Hamiltonian takes the form

$$H^* = \omega_1 J_1 + \omega_2 J_2 + \Omega J_3 + \varepsilon^{2/3} \sum_{i, j = 1;\, i \le j}^{3} c_{ij} J_i J_j + \varepsilon^{1/3} H_0^{*(1)} + \varepsilon^{2/3} H_1^{*(1)} + O(\varepsilon) \tag{4.1}$$

The function $H_0^{*(1)}$ is obtained from the function $H_0^{(1)}$, which has been defined in (2.3), by replacing φ by ψ_3, while the function $H_1^{*(1)}$ is obtained from the function (3.2) by replacing Q by ψ_3, and Q_i and P_i $(i = 1, 2)$ by $\sqrt{2J_i}\sin\psi_i$ and $\sqrt{2J_i}\cos\psi_i$.

Using the canonical replacement

$$J_3^* = J_3 - \varepsilon^{1/3}\frac{H_0^{*(1)}}{\Omega}, \quad \psi_3 = \psi_3$$

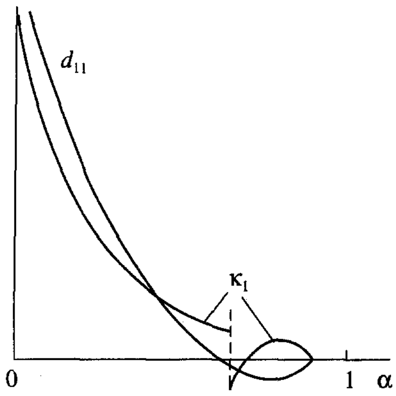

Fig. 5

we eliminate the term with $H_0^{*(1)}$ in (4.1), and, in $H_1^{*(1)}$, we also eliminate all terms with non-resonance harmonics.

4.1. The case when $\omega_1 \simeq -2\Omega$

Initially suppose $\omega_1 \simeq -2\Omega$. Then, the term $H_1^{*(1)}$ remains in $\kappa_1\sqrt{J_1}\sin(\psi_1 + 2\psi_3)$, where $\kappa_1 = \sqrt{2}\,(a_1 + 2b_1)/4$, (the quantities a_1 and b_1 are defined in (3.2)).

We now carry out the canonical replacement of variables.

$$\psi_1 = \psi_1^* - 2\psi_3^* + \frac{\pi}{2}, \quad \psi_2 = \psi_2^*, \quad \psi_3 = \psi_3^*$$

$$J_1 = I_1, \quad J_2 = I_2, \quad J_3 = I_3 + 2I_1$$

and introduce the resonance detuning by putting $\omega_1/\Omega = -2 + \varepsilon^{2/3}\delta$. The Hamiltonian of the system takes the form

$$\begin{aligned} H &= \Omega I_3 + \omega_2 I_2 + \varepsilon^{2/3}\{(d_{22}I_2^2 + d_{23}I_2I_3 + d_{33}I_3^2) + \\ &+ [(\delta\Omega + d_{12}I_2 + d_{13}I_3)I_1 + \kappa_1\sqrt{I_1}\cos\psi_1^* + d_{11}I_1^2]\} + O(\varepsilon) \qquad (4.2)\\ d_{11} &= c_{11} + 2c_{13} + 4c_{33}, \quad d_{12} = c_{12} + 2c_{23}, \quad d_{13} = c_{13} + 4c_{33}, \quad d_{jk} = c_{jk}, \quad j, k = 2, 3 \end{aligned}$$

The values of the frequencies Ω and ω_2 in (4.2) must be calculated for values of the parameters α and θ_0 belonging to the resonance curve $\omega_1 = -2\Omega$ or a small neighbourhood of this curve. It suffices to calculate the values of the coefficients κ_1 and d_{ij} for points $(\alpha, \theta_0(\alpha))$ lying on the resonance curve.

The form of the periodic solution is determined by the values of the coefficients κ_1 and d_{11}. The relations $\kappa_1 = \kappa_1(\alpha)$ and $d_{11} = d_{11}(\alpha)$ ($\alpha \in (0, 0.900)$) on the resonance curve $\omega_1 = -2\Omega$ are shown in Fig. 5.

We exclude the points $\alpha = 0.764\ldots$, $\alpha = 0.626\ldots$ (the zeros of these functions) from consideration. At the point of discontinuity $\alpha = 2/3$ on the graph of the function $\kappa_1 = \kappa_1(\alpha)$, the coefficients n_{ij} of the linear replacement of variables (2.4) have to be calculated using formulae (2.6). All the remaining transformations and calculations are carried out as in the general case, and we obtain that $\kappa_1 \neq 0$.

We make the following replacement of variables

$$I_i = \kappa_{1*}\rho_i, \quad i = 1, 2, 3, \quad \psi_1^* = \theta_1 + \sigma_1, \quad \psi_j^* = \theta_j, \quad j = 2, 3$$

$$\kappa_{1*} = (\kappa_1/d_{11})^{2/3}, \quad \sigma_1 = \pi(1 - \mathrm{sign}(\kappa_1 d_{11}))/2$$

The transformed Hamiltonian takes the following final form

$$\begin{aligned} \hat{H} &= \Omega\rho_3 + \omega_2\rho_2 + \varepsilon^{2/3}\{(\alpha_1\rho_2^2 + \alpha_2\rho_2\rho_3 + \alpha_3\rho_3^2) + \\ &+ \beta[(b_1\delta + b_2\rho_2 + b_3\rho_3)\rho_1 + \rho_1^2 + \sqrt{\rho_1}\cos\theta_1]\} + O(\varepsilon) \\ \alpha_1 &= d_{22}\kappa_{1*}, \quad \alpha_2 = d_{23}\kappa_{1*}, \quad \alpha_3 = d_{33}\kappa_{1*}, \quad \beta = d_{11}\kappa_{1*} \\ b_1 &= \Omega/\beta, \quad b_2 = d_{12}/d_{11}, \quad b_3 = d_{13}/d_{11} \end{aligned} \tag{4.3}$$

The term $O(\varepsilon)$ in (4.3) is π-periodic in θ_3.

We will now construct the periodic motions of the system with the Hamiltonian (4.3) (see [1]) and we will initially consider the motions of the system on an isoenergetic level. Using the energy integral $\hat{H} = \Omega h = \text{const}$, we change to a reduced system with two degrees of freedom with the independent variable θ_3. The Hamiltonian of this system has the form

$$\begin{aligned} K &= \frac{\omega_2}{\Omega}\rho_2 + \frac{\varepsilon^{2/3}}{\Omega}\left\{\left[\alpha_1\rho_2^2 + \alpha_2\left(h - \frac{\omega_2}{\Omega}\rho_2\right)\rho_2 + \alpha_3\left(h - \frac{\omega_2}{\Omega}\rho_2\right)^2\right] + \right. \\ &\left. + \beta\left[\left(b_1\delta + b_2\rho_2 + b_3\left(h - \frac{\omega_2}{\Omega}\rho_2\right)\right)\rho_1 + \rho_1^2 + \sqrt{\rho_1}\cos\theta_1\right]\right\} + O(\varepsilon) \end{aligned} \tag{4.4}$$

We will also consider the approximate Hamiltonian $\hat{K}$, which is obtained from K by neglecting the term $O(\varepsilon)$. In the system with the Hamiltonian $\hat{K}$, the coordinate θ_2 is cyclic and hence $\rho_2 = c_2 = \text{const}$. If the additive constant is discarded, the Hamiltonian $\hat{K}$ takes the form

$$\begin{aligned} \hat{K} &= \frac{\varepsilon^{2/3}\beta}{\Omega}H', \quad H' = -\chi\rho_1 + \rho_1^2 + \sqrt{\rho_1}\cos\theta_1 \\ \chi &= -\left[b_1\delta + b_2c_2 + b_3\left(h - \frac{\omega_2}{\Omega}c_2\right)\right] = \text{const} \end{aligned} \tag{4.5}$$

The function H' is a model Hamiltonian for systems with one degree of freedom in the case of resonance in the forced oscillations [13]. However, if the parameter χ is solely determined by the magnitude of the resonance detuning in these systems, then, in the Hamiltonian H', the parameter χ still depends on the constant c_2 of the cyclic integral (which is associated with the existence of the cyclic coordinate θ_2) and on the constant energy h of the initial system with three degrees of freedom.

The equilibrium positions of the approximate system with Hamiltonian (4.5) are described by the relations

$$\rho_2 = c_2 = 0, \quad \rho_1 = \rho_{1*}, \quad \theta_1 = \theta_{1*} \tag{4.6}$$

where θ_{1*}, ρ_{1*} is one of the equilibrium positions of the model system with the Hamiltonian H'. When $\chi < 3/2$, the model system has a single stable equilibrium position

$$\rho_1^{(0)} = \frac{|\chi|}{3}\operatorname{ch}\frac{\varphi}{3} + \frac{\chi}{3}, \quad \theta_1^{(0)} = \pi \quad \left(\operatorname{ch}\varphi = \frac{27 - 4\chi^3}{4|\chi|^3}\right) \tag{4.7}$$

and, when $\chi \geqslant 3/2$, it has three equilibrium positions

$$\begin{aligned} \rho_1^{(1)} &= -\frac{\chi}{3}\cos\frac{\varphi}{3} + \frac{\chi}{3}, \quad \theta_1^{(1)} = 0; \quad \rho_1^{(2)} = -\frac{\chi}{3}\cos\left(\frac{\varphi}{3} + \frac{4\pi}{3}\right) + \frac{\chi}{3}, \quad \theta_1^{(2)} = 0 \\ \rho_1^{(3)} &= -\frac{\chi}{3}\cos\left(\frac{\varphi}{3} + \frac{2\pi}{3}\right) + \frac{\chi}{3}, \quad \theta_1^{(3)} = \pi \quad \left(\cos\varphi = \frac{4\chi^3 - 27}{4\chi^3}\right) \end{aligned} \tag{4.8}$$

of which two, that correspond to the greatest ($\rho_1^{(3)}$) and smallest ($\rho_1^{(1)}$) values of ρ_1, are stable, while the one corresponding to the middle value of ρ_1 is unstable [13].

When $\chi = 3/2$, the model system has an unstable complex singular point $\rho_1 = 1/4$, $\theta_1 = 0$, which we shall subsequently exclude from considerations and a stable equilibrium position $\rho_1 = 1$, $\theta_1 = \pi$.

In the neighbourhood of the equilibrium position (4.6) of the approximation system, the complete Hamiltonian (4.4) of the reduced system has the form (it is assumed that $\rho_2 = r_2$, $\theta_1 = \theta_{1*} + x_1$, $\rho_1 = \rho_{1*} + y_1$)

$$K = \left[\frac{\omega_2}{\Omega} + O(\varepsilon^{2/3})\right]r_2 + \frac{\varepsilon^{2/3}\beta}{\Omega}\left[\rho_{1*}(2\rho_{1*} - \chi)x_1^2 + \frac{6\rho_{1*} - \chi}{4\rho_{1*}}y_1^2 + O_3\right] + O(\varepsilon) \tag{4.9}$$

where O_3 is the set of terms, the power of which with respect to x_1, y_1, $r_2^{1/2}$ is no lower than the third. The term $O(\varepsilon)$ in (4.9) is π-periodic in θ_3.

Calculation showed that the quantity ω_2/Ω is not close to an even integer for values of the parameters α and θ_0 belonging to the resonance curve $\omega_1 = -2\Omega$ and a small neighbourhood of this curve. The non-resonance case in Poincaré's theory of periodic motions holds and, from each equilibrium position of the approximate system with the Hamiltonian $\hat{K}$, a unique solution of the system with the Hamiltonian K is generated, which is π-periodic in θ_3 and analytic in $\varepsilon^{1/3}$, and this solution has the form

$$\theta_1 = \tilde{\theta}_1(\theta_3) + \theta_{1*} + O(\varepsilon), \quad \rho_1 = \tilde{\rho}_1(\theta_3) = \rho_{1*} + O(\varepsilon), \quad \rho_2 = \tilde{\rho}_2(\theta_3) = O(\varepsilon^2) \tag{4.10}$$

The last relation in (4.10) can be replaced by two relations for the Cartesian coordinates θ_2, ρ_2 corresponding to the pair x_2, y_2 ($x_2 = \sqrt{2\rho_2}\sin\theta_2$, $y_2 = \sqrt{2\rho_2}\cos\theta_2$

$$x_2 = \tilde{x}_2(\theta_3) = O(\varepsilon), \quad y_2 = \tilde{y}_2(\theta_3) = O(\varepsilon)$$

From the relations (4.10) and the energy integral of the system with three degrees of freedom with Hamiltonian (4.3), we obtain

$$\rho_3 = \tilde{\rho}_3(\theta_3) = h - \frac{\varepsilon^{2/3}}{\Omega}\{\alpha_3 h^2 + \beta[-\chi\rho_{1*} + \rho_{1*}^2 + \sqrt{\rho_{1*}}\cos\theta_{1*}]\} + O(\varepsilon^{5/3})$$
$$\chi = -(b_1\delta + b_3 h) \tag{4.11}$$

where the term $O(\varepsilon^{5/3})$ is π-periodic in θ_3.

Relations (4.10) and (4.11) describe a one-parameter family of solutions (the energy constant h acts as the parameter) of the system with three degrees of freedom with Hamiltonian (4.3) which are π-periodic in θ_3 and analytic in the quantity $\varepsilon^{1/3}$. There are one or three such families depending on the parameter χ of the model system.

In the initial variables, we have the following family of motions of the satellite, which is π-periodic in φ (the angle of natural rotation) and analytic in $\varepsilon^{1/3}$

$$\theta = \theta_0 + \varepsilon^{1/3}n_{11}\sqrt{2\kappa_{1*}\rho_{1*}}\cos(\theta_{1*} + \sigma_1 - 2\varphi) + O(\varepsilon^{2/3})$$
$$\psi = -\varepsilon^{1/3}n_{23}\sqrt{2\kappa_{1*}\rho_{1*}}\sin(\theta_{1*} + \sigma_1 - 2\varphi) + O(\varepsilon^{2/3}) \tag{4.12}$$

Here, the change in the variable φ with τ is described by the equation

$$\frac{d\varphi}{d\tau} = \Omega + \varepsilon^{1/3}g_1(\varphi)$$

$$g_1(\varphi) = (m_1 n_{11} + m_2 n_{41})\sqrt{2\kappa_{1*}\rho_{1*}}\cos(\theta_{1*} + \sigma_1 - 2\varphi) + O(\varepsilon^{1/3})$$

The period of the motions (4.12) with respect to τ is equal to

$$T = 2\pi/\Omega^*, \quad \Omega^* = \Omega + \varepsilon^{1/3}\bar{g}_1 = \Omega + O(\varepsilon^{2/3})$$

where $\bar{g}_1$ is the mean value of the function $g_1(\varphi)$ over a period π.

If the terms $O(\varepsilon^{2/3})$ are neglected, relations (4.12) determine the motion of the satellite when the unit vector of its axis describes a closed, three-dimensional curve on a unit sphere with its centre at the centre of mass of the satellite, the projection of which onto the $GX'Y'$ plane (which is described in Section 3.2) is the ellipse

$$\frac{X'^2}{(\varepsilon^{1/3}\sin\theta_0 n_{23}\sqrt{2\kappa_{1*}\rho_{1*}})^2}+\frac{Y'^2}{(\varepsilon^{1/3}n_{11}\sqrt{2\kappa_{1*}\rho_{1*}})^2} = 1$$

with semi-axes of the order of $\varepsilon^{1/3}$.

We will consider the problem of the stability of the periodic motions which have been found. Motions generated from an unstable equilibrium position of the model system with the Hamiltonian H' will be unstable, since the characteristic equation of the linearized, approximate system has a positive real root.

In order to solve the problem of the stability of the motions generated from the stable equilibrium positions of the model system, we specify the perturbations of the variables of the system with Hamiltonian (4.3) relative to their values for the periodic motions, using the following canonical transformation (which is analogous to (3.6))

$$\theta_1 = \tilde{\theta}_1 + X_1,\quad \rho_1 = \tilde{\rho}_1 + Y_1,\quad x_2 = \tilde{x}_2 + X_2,\quad y_2 = \tilde{y}_2 + Y_2,\quad \theta_3 = w$$

$$\rho_3 = \tilde{\rho}_3 + X_1\frac{d\tilde{\rho}_1}{dw} - Y_1\frac{d\tilde{\theta}_1}{dw} + X_2\frac{d\tilde{y}_2}{dw} - Y_2\frac{d\tilde{x}_2}{dw} + r_3^*$$

We then normalize the perturbed Hamiltonian up to terms of the fourth order inclusive. In the "polar" coordinates φ_i, r_i $(i = 1, 2, 3)$, the normalized Hamiltonian takes the form

$$H = \varepsilon^{2/3}\beta\omega^* r_1 + \omega_2^* r_2 + \Omega^* r_3 + \varepsilon^{2/3}\sum e_{ij}r_ir_j + O(\varepsilon)$$

where

$$\omega^* = \omega + O(\varepsilon^{1/3})(\omega^2 = (6\rho_{1*}-\chi)(2\rho_{1*}-\chi)),\quad \omega_2^* = \omega_2 + O(\varepsilon^{2/3}),\quad \Omega^* = \Omega + O(\varepsilon^{2/3})$$

and the coefficients e_{ij} (if the terms in them of the order of $\varepsilon^{1/3}$ and higher are neglected) are calculated using the formulae

$$e_{11} = \frac{\beta}{4}\left[2(a_6+3a_5+3a_7) - \frac{3a_3^2+6a_3a_4+15a_4^2}{\omega}\right],\quad e_{12} = -\frac{\beta c_1}{\omega}(a_3+3a_4)$$

$$e_{13} = -\frac{\beta c_2}{\omega}(a_3+3a_4),\quad e_{22} = \alpha_1 - \frac{\beta c_1^2}{2\omega},\quad e_{23} = \alpha_2 - \frac{\beta c_1 c_2}{\omega},\quad e_{33} = \alpha_3 - \frac{\beta c_2^2}{2\omega}$$

$$a_3 = -\frac{\chi-2\rho_{1*}}{2}s,\quad a_4 = \frac{\chi-2\rho_{1*}}{8\rho_{1*}^2 s^3},\quad a_5 = \frac{\rho_{1*}(\chi-2\rho_{1*})}{12}s^4$$

$$a_6 = \frac{\chi-2\rho_{1*}}{8\rho_{1*}},\quad a_7 = -\frac{5(\chi-2\rho_{1*})}{64\rho_{1*}^3 s^4},\quad c_1 = \frac{b_2}{s},\quad c_2 = \frac{b_3}{s}$$

$$s = \left[\frac{6\rho_{1*}-\chi}{4\rho_{1*}^2(2\rho_{1*}-\chi)}\right]^{1/4}$$

The coefficients e_{ij} depend on the parameter χ of the model system and also on the position of the point $(\alpha, \theta_0(\alpha))$ on the resonance curve.

We now check the conditions, analogous to (3.10), for all permissible values of the parameters χ and α. Calculations show that, in the case of the periodic motions which are generated, when $\chi > 3/2$, from the stable equilibrium positions of the model system to which $\rho_{1*} = \rho_1^{(1)}$ or $\rho_{1*} = \rho_1^{(3)}$ correspond, the above-mentioned conditions are satisfied and these motions are orbitally stable for the majority of initial conditions. In the case when $\chi < 3/2$ and of the periodic motions generated from the stable equilibrium

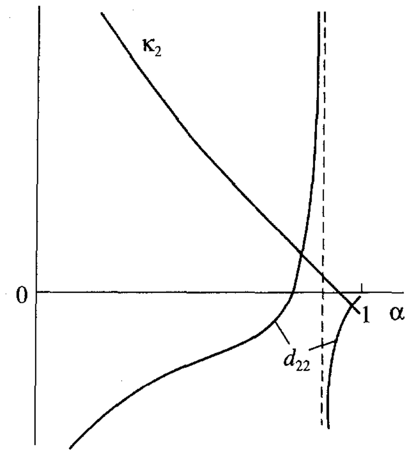

Fig. 6

position of the model system to which $\rho_{1*} = \rho_1^{(0)}$ corresponds, these conditions are violated at the single point $\alpha = 0.626 \ldots$, $\chi = -0.459 \ldots$, where $D_3 = D_4 = 0$. In the case of the remaining permissible values of the parameters α and χ, the motions being considered are orbitally stable for the majority of initial conditions.

4.2. The case when $\omega_2 \simeq -2\Omega$

The periodic motions of a satellite which arise in the neighbourhood of its conical precession in the case of the resonance $\omega_2 \simeq -2\Omega$ can be obtained in a similar manner.

A denumerable set of points exists on the resonance curve $\omega_2 \simeq -2\Omega$ for which we have $\omega_1/\omega_2 = N$ ($N = 2, 3, \ldots$) (the points of intersection of the curve $\omega_2 = -2\Omega$ with the curves $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_2$ correspond to cases $N = 2$ and $N = 3$ respectively, see Fig. 1). These cases of multiple resonance are not considered further.

Outside the above-mentioned points and their small neighbourhoods, the resonance coefficient $\kappa_2 = \sqrt{2}\,(a_2 + 2b_2)/4$ and the coefficient $d_{22} = c_{22} + 2c_{23} + 4c_{33}$ in the fourth-order terms of the Hamiltonian, which is analogous to (4.2), determine the form of the periodic motions.

Graphs of the functions $\kappa_2 = \kappa_2(\alpha)$ and $d_{22} = d_{22}(\alpha)$ for the resonance curve $\omega_2 = -2\Omega$ are shown in Fig. 6. These functions have zeros at $\alpha = 0.925 \ldots$ and $\alpha = 0.844 \ldots$ respectively. Moreover, the function $d_{22} = d_{22}(\alpha)$ has a discontinuity at $\alpha = 0.888 \ldots$ (at the point of intersection of the resonance curves $\omega_2 = -2\Omega$ and $\omega_1 = 2\omega_2$). We also exclude these values of α from consideration.

For the remaining values of α we obtain the families of motions of the satellite

$$\begin{aligned}
\theta &= \theta_0 + \varepsilon^{1/3} n_{12}\sqrt{2\kappa_{2*}\rho_{2*}}\cos(\theta_{2*} + \sigma_2 - 2\varphi) + O(\varepsilon^{2/3}) \\
\psi &= -\varepsilon^{1/3} n_{24}\sqrt{2\kappa_{2*}\rho_{2*}}\sin(\theta_{2*} + \sigma_2 - 2\varphi) + O(\varepsilon^{2/3}) \qquad (4.13) \\
\kappa_{2*} &= (\kappa_2/d_{22})^{2/3}, \quad \sigma_2 = \pi(1 - \mathrm{sign}(\kappa_2 d_{22}))/2
\end{aligned}$$

which are π-periodic in φ and analytic in $\varepsilon^{1/3}$.

Here, (θ_{2*}, ρ_{2*}) are the equilibrium positions of the model system, defined by formulae (4.7) and (4.8). The parameter χ of the model system for the periodic solutions (4.13) is calculated using the formula

$$\begin{aligned}
&\chi = -(\tilde{b}_1\tilde{\delta} + \tilde{b}_3 h) \\
&\tilde{b}_1 = \Omega/\tilde{\beta}, \quad \tilde{\beta} = \tilde{d}_{22}\kappa_{2*}, \quad \tilde{b}_3 = \tilde{d}_{23}/\tilde{d}_{22}, \quad \tilde{d}_{22} = c_{22} + 2c_{23} + 4c_{33}, \quad \tilde{d}_{23} = c_{23} + 4c_{33}
\end{aligned}$$

where $\tilde{\delta}$ is the resonance detuning, introduced using the formula $\omega_2/\Omega = -2 + \varepsilon^{2/3}\tilde{\delta}$.

In the solutions (4.13), the change in the variable φ with τ is described by the equation

$$\frac{d\varphi}{d\tau} = \Omega + \varepsilon^{1/3} g_2(\varphi)$$

$$g_2(\varphi) = (m_1 n_{12} + m_2 n_{42})\sqrt{2\kappa_{2*}\rho_{2*}}\cos(\theta_{2*} + \sigma_2 - 2\varphi) + O(\varepsilon^{1/3})$$

The period of the motion (4.13) with respect to τ is equal to

$$T = 2\pi/\Omega_2^*, \quad \Omega_2^* = \Omega + \varepsilon^{1/3}\bar{g}_2 = \Omega + O(\varepsilon^{2/3})$$

where $\bar{g}_2$ is the mean value of the function $g_2(\varphi)$ over a period π.

Depending on the parameter χ of the model system, there is one or three periodic families of the form of (4.13).

The motions of a satellite which are generated (when $\chi > 3/2$) from an unstable equilibrium position of the model system will be unstable. Motions generated from stable equilibrium positions of the model system are orbitally stable for the majority of initial conditions since, as calculations show, conditions similar to (3.10) are satisfied for all permissible values of the parameters α and χ.

This research was supported financially by the Russian Foundation for Basic Research (02-01-00831).

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Translated by E.L.S